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HOW TO COMPUTE THE MULTIGRADED HILBERT DEPTH OF A MODULE

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Abstract. The aim of this paper is to introduce a method for computing Hilbert decompositions (and consequently the Hilbert depth) of a finitely generated multigraded module M over the polynomial ring $K[X_1, \dots, X_n]$ by reducing the problem to the computation of the finite set of the new defined Hilbert partitions. Moreover, in the last part we show that Hilbert partitions may also be used for computing the Stanley depth of the module M.

1. Introduction

In recent years *Stanley decompositions* of multigraded modules over standard multigraded polynomial rings $R = K[X_1, \dots, X_n]$ have been discussed intensively. Such decompositions, introduced by Stanley in [15], break the module M into a direct sum of *Stanley spaces*, each being of type mS where m is a homogeneous element of M, $S = K[X_{i_1}, \dots, X_{i_d}]$ is a polynomial subalgebra of R and $S \bigcap \operatorname{Ann} m = 0$. One says that M has *Stanley depth* s, $\operatorname{Stdepth} M = s$, if one can find a Stanley decomposition in which $d \geq s$ for each polynomial subalgebra involved, but none with s replaced by $s + 1$. Stanley conjectured (even for the case of arbitrary gradings) that

$$\operatorname{Stdepth} M \geq \operatorname{depth} M.$$

The computation of the Stanley depth is not an easy task. Some years ago, Herzog, Vladoiu and Zheng introduced in [11] a method for computing the Stanley depth of a monomial ideal. Further, they proved that $\operatorname{Stdepth} \mathfrak{m} = \lfloor n/2 \rfloor$ for $n \leq 9$, where $\mathfrak{m} = (X_1, \dots, X_n)$ is the multigraded maximal ideal of R, and conjectured that it holds for all n (cf. [11, p. 3152]).

This conjecture was positively answered in a paper of Biró, Howard, Keller, Trotter and Young in 2010 by using (only) combinatorial methods [4]. Other remarkable results in the study of the Stanley depth in the multigraded case were presented by Apel (see [1], [2]), Herzog et al. (see [9], [10]) and Popescu et al. (see [3], [14]).

Later, a new type of decompositions for multigraded modules was introduced by Bruns, Uliczka and Krattenthaler in [6]. The Stanley decomposition is replaced by the *Hilbert decomposition*, which is a weaker type of decomposition not requiring the

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summands to be submodules of M, but only vector subspaces isomorphic to polynomial subrings. Its name comes from the fact that it depends only on the Hilbert series of M. The *Hilbert depth* $\operatorname{Hdepth} M$ is defined accordingly. Since all Stanley decompositions are Hilbert decompositions, the latter are prerequisites to the existence of Stanley decompositions, and since they are easier to find, one may try to construct a Stanley decomposition by converting a "good" Hilbert decomposition.

As a consequence, several results concerning both the graded and multigraded cases were presented in [7], [16] and [12]. All of them are strongly based on combinatorial techniques.

The aim of this paper is to provide a method to compute the Hilbert depth of a finitely generated multigraded module M over the polynomial ring R. The procedure presented in Section 3 for computing the Hilbert depth can be seen as a natural generalization of the method introduced in [11] for computing the Stanley depth of the monomial ideals of R—remark that the Hilbert depth coincides with the Stanley depth in this case—but in the general case of multigraded R-modules this cannot be easily extended. By using the new concept of Hilbert partition (cf. Definition 3.1) and the functorial techniques exposed by E. Miller in [13], we obtain a method for computing the Hilbert depth in the general case of a multigraded R-module (see Theorem 3.3 and Corollary 3.4).

In Section 4 we present an approach to the problem of computing Stanley depth of a finitely generated multigraded module M over the polynomial ring R based on Section 3. We show that in a *finite* number of steps one can decide whether a Hilbert partition is inducing a Stanley decomposition or not (the converse is always true: any Stanley decomposition induces a Hilbert partition), see Proposition 4.4. We conclude that one may compute the Stanley depth by looking at all the Hilbert partitions and selecting those that are also inducing Stanley decompositions (Corollary 4.7).

In the last section we show that the methods introduced in the Sections 3 and 4 can effectively be used in order to deduce some simple statements.

2. Prerequisites

Thorough the paper we will use the notation $a = (a_1, \ldots, a_n)$ for elements $a \in \mathbb{Z}^n$ (or $\mathbb{N}^n$). We consider the polynomial ring $R = K[X_1, \ldots, X_n]$ over a field K with the *multigraded* structure on R, namely the $\mathbb{Z}^n$-grading in which the degree of X_i is the i-th vector e_i of the canonical basis of $\mathbb{R}^n$. For any $c \in \mathbb{N}^n$ we will denote $X^c := X_1^{c_1} X_2^{c_2} \cdots X_n^{c_n}$. All R-modules we consider are assumed to belong to the category $\mathcal{M}$ of finitely generated $\mathbb{Z}^n$-graded (or multigraded) R-modules.

Hilbert functions are the most important numerical invariants of graded and multigraded modules; they form the bridge from commutative algebra to its combinatorial applications. Let $M = \bigoplus_{a \in \mathbb{Z}^n} M_a \in \mathcal{M}$. Then we can consider its *Hilbert function*

$$H(M, -) : \mathbb{Z}^n \longrightarrow \mathbb{Z}, \quad H(M, a) = \dim_K M_a.$$

For further details about Hilbert functions in the multigraded case the reader is referred to [5].

From the combinatorial viewpoint a module is often only an algebraic substrate of its Hilbert function, and one may ask which presentation a given Hilbert function can have. Following [6] we define the main objects of our study, namely Hilbert decompositions and Hilbert depth of modules.

Definition 2.1. A *Hilbert decomposition of* M is a finite family

$$\mathfrak{D} : (R_i, s_i)_{i\in I}$$

such that R_i are subalgebras generated by a subset of the indeterminates of R for each $i \in I$, $s_i \in \mathbb{Z}^n$, and

$$M \cong \bigoplus_{i\in I} R_i(-s_i)$$

as a multigraded K-vector space.

Observe that all the Hilbert decompositions of a module M depend only on the Hilbert function of M.

Definition 2.2. A Hilbert decomposition carries the structure of an R-module and has a well-defined depth, which is called the *depth of the Hilbert decomposition* $\mathfrak{D}$ and will be denoted by $\operatorname{depth}\mathfrak{D}$. The *Hilbert depth* of a module M is

$$\max\{\operatorname{depth}\mathfrak{D} \mid \mathfrak{D} \text{ is a Hilbert decomposition of } M\}$$

and will be denoted by $\operatorname{Hdepth} M$.

Next, we shall consider a natural partial order on $\mathbb{Z}^n$ as follows: Given $a, b \in \mathbb{Z}^n$, we say that $a \preceq b$ if and only if $a_i \leq b_i$ for $i = 1, \ldots, n$. Note that $\mathbb{Z}^n$ with this partial order is a distributive lattice with meet $a \wedge b$ and join $a \vee b$ being the componentwise minimum and maximum, respectively. We set the interval between a and b to be

$$[a, b] = \{c \in \mathbb{Z}^n \mid a \preceq c \preceq b\}.$$

We recall some definitions and results given by Ezra Miller in [13] which will be useful in the sequel. Let $g \in \mathbb{N}^n$. The module M is said to be $\mathbb{N}^n$-graded if $M_a = 0$ for $a \notin \mathbb{N}^n$; M is said to be positively g-determined if it is $\mathbb{N}^n$-graded and the multiplication map $\cdot X_i : M_a \longrightarrow M_{a+e_i}$ is an isomorphism whenever $a_i \geq g_i$. A characterization of positively g-determined modules is given by the following.

Proposition 2.3. [13, Proposition 2.5] *The module $M \in \mathcal{M}$ is positively g-determined if and only if the multigraded Betti numbers of M satisfy $\beta_{0,a} = \beta_{1,a} = 0$ unless $0 \preceq a \preceq g$.*

Important tools also introduced in [13, Definition 2.7] are the following functors:

(1) the subquotient bounded in the interval $[0, g]$, denoted by $\mathcal{B}_g$, where

$$\mathcal{B}_g(M) := \bigoplus_{0\preceq a\preceq g} M_a;$$

(2) the positive extension of M, denoted by $\mathcal{P}_g$, where

$$\mathcal{P}_g(M) := \bigoplus_{a \in \mathbb{Z}^n} M_{g \wedge a}$$

or, in other words, $(\mathcal{P}_g M)_a = M_{g \wedge a}$, endowed with the R-action

$$(\cdot X_i)_g : (\mathcal{P}_g M)_a \to (\mathcal{P}_g M)_{a+e_i}$$

defined as the multiplication map $\cdot X_i : M_{g \wedge a} \to M_{g \wedge a + e_i}$ if $a_i < g_i$; or as the identity map otherwise.

From the above definitions one can immediately obtain:

Proposition 2.4. [13, Theorem 2.11] *Assume that $M \in \mathcal{M}$ is positively g-determined. Then $\mathcal{P}_g(\mathcal{B}_g(M)) = M$.*

The following example makes clear the behaviour of the functors $\mathcal{B}_g$ and $\mathcal{P}_g$.

Example 2.5. [13, Example 2.8] Let $a \in \mathbb{N}^n$. We have $\mathcal{B}_g(R(-a)) = 0$ unless $a \preceq g$, in which case we have that

$$\mathcal{B}_g(R(-a)) \cong (R/\langle X_1^{g_1+1-a_1}, \ldots, X_n^{g_n+1-a_n} \rangle)(-a)$$

is the artinian subquotient of R which is nonzero precisely in the degrees from the interval $[a, g]$. Applying $\mathcal{P}_g$ to this yields back $R(-a)$ so that $\mathcal{P}_g(\mathcal{B}_g(R(-a)))$ is isomorphic to $R(-a)$ if $a \preceq g$.

Figure 1, in which the circles represent the graded components of $\mathcal{B}_g(M)$, illustrates the action of the functor $\mathcal{P}_g$.

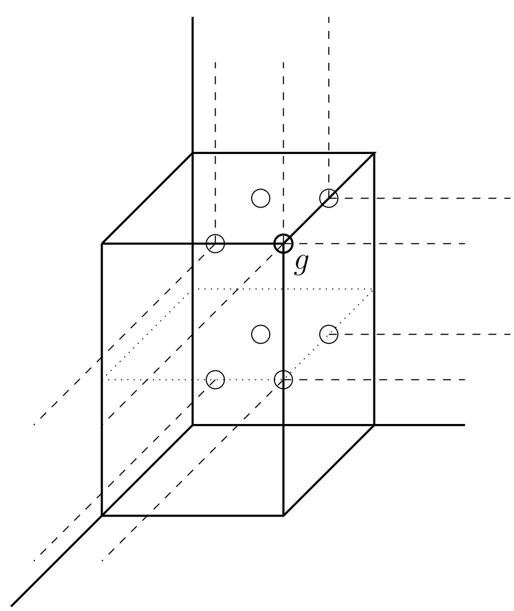

FIGURE 1. The functor $\mathcal{P}_g$

3. A method for computing the multigraded Hilbert depth of a module

The aim of this section is to describe a procedure for computing the Hilbert depth of a multigraded module over the polynomial ring. Let M denote a finitely generated multigraded R-module with a minimal multigraded free presentation

$$\bigoplus_{a\in\mathbb{Z}^n} R(-a)^{\beta_{1,a}} \longrightarrow \bigoplus_{a\in\mathbb{Z}^n} R(-a)^{\beta_{0,a}} \longrightarrow M \longrightarrow 0,$$

and assume for simplicity, and without loss of generality, that all $\beta_{0,a} = 0$ (and *a fortiori* all $\beta_{1,a} = 0$) if $a \notin \mathbb{N}^n$.

We shall also consider the *Hilbert series* of M, that is

$$H_M(X) = \sum_{a\in\mathbb{N}^n} H(M,a)X^a.$$

Let $g \in \mathbb{N}^n$ be such that the multigraded Betti numbers of M satisfy the equalities $\beta_{0,a} = \beta_{1,a} = 0$ unless $0 \preceq a \preceq g$. Then, according to Proposition 2.3, the module M is positively g-determined. The Hilbert series of M can be recovered from the polynomial

$$H_M(X)_{\preceq g} := \sum_{0\preceq a\preceq g} H(M,a)X^a = H_{\mathcal{B}_g(M)}(X)$$

since, according to Proposition 2.4, we have that $\mathcal{P}_g(\mathcal{B}_g(M)) = M$. This fact may be used in order to actually compute the Hilbert depth of M, as presented in this section.

Given $a, b \in \mathbb{Z}^n$ such that $a \preceq b$, we set

$$Q[a,b](X) := \sum_{a\preceq c\preceq b} X^c$$

and call it the *polynomial induced by the interval* $[a, b]$.

Definition 3.1. We define a *Hilbert partition* of the polynomial $H_M(X)_{\preceq g}$ to be an expression

$$\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i\in I_{\mathfrak{P}}} Q[a^i, b^i](X)$$

as a finite sum of polynomials induced by the intervals $[a^i, b^i]$ (the notation $I_{\mathfrak{P}}$ makes clear the dependency on $\mathfrak{P}$ and so the finiteness).

In order to describe the Hilbert decomposition of M induced by the Hilbert partition $\mathfrak{P}$ of $H_M(X)_{\preceq g}$, we introduce the following notations. For $a \preceq g$ we set $Z_a = \{X_j \mid a_j = g_j\}$. Moreover we denote by $K[Z_a]$ the subalgebra generated by the subset of the indeterminates Z_a. We also define the map

$$\rho : \{0 \preceq a \preceq g\} \longrightarrow \mathbb{N}, \quad \rho(a) := |Z_a|,$$

and for $0 \preceq a \preceq b \preceq g$ we set

$$\mathcal{G}[a,b] = \{c \in [a,b] \mid c_j = a_j \text{ for all } j \in \mathbb{N} \text{ with } X_j \in Z_b\}.$$

Lemma 3.2. *Let $0 \preceq a \preceq b \preceq g$. Set $K[a,b] = \mathcal{B}_b(R(-a))$. Then*

$$\mathcal{P}_g(K[a,b]) = \bigoplus_{c \in \mathcal{G}[a,b]} K[Z_b](-c)$$

is a Hilbert decomposition of $\mathcal{P}_g(K[a,b])$.

Proof. We have

$$\mathcal{B}_b(R(-a)) = \mathcal{B}_b(\bigoplus_{c \in \mathcal{G}[a,b]} K[Z_b](-c)) = \mathcal{B}_g(\bigoplus_{c \in \mathcal{G}[a,b]} K[Z_b](-c)).$$

Since $\mathcal{B}_g$ and $\mathcal{P}_g$ are K-linear functors, the conclusion follows from Proposition 2.4 by applying $\mathcal{P}_g$. □

We can now state the main theorem of this paper.

Theorem 3.3. *The following statements hold:*

(1) *Let $\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i=1}^r Q[a^i, b^i](X)$ be a Hilbert partition of $H_M(X)_{\preceq g}$. Then*

$$\mathfrak{D}(\mathfrak{P}) : M = \bigoplus_{i=1}^r \Big(\bigoplus_{c \in \mathcal{G}[a^i,b^i]} K[Z_{b^i}](-c) \Big) \qquad [\star]$$

is a Hilbert decomposition of M. Moreover,

$$\operatorname{Hdepth} \mathfrak{D}(\mathfrak{P}) = \min\{\rho(b^i) : \ i = 1, \ldots, r\}.$$

(2) *Let $\mathfrak{D}$ be a Hilbert decomposition of M. Then there exists a Hilbert partition $\mathfrak{P}$ of $H_M(X)_{\preceq g}$ such that*

$$\operatorname{Hdepth} \mathfrak{D}(\mathfrak{P}) \geq \operatorname{Hdepth} \mathfrak{D}.$$

In particular, $\operatorname{Hdepth} M$ *can be computed as the maximum of the numbers* $\operatorname{Hdepth} \mathfrak{D}(\mathfrak{P})$*, where $\mathfrak{P}$ runs over the finitely many Hilbert partitions of $H_M(X)_{\preceq g}$.*

Proof. (1) The Hilbert partition $\mathfrak{P}$ of $H_M(X)_{\preceq g}$ induces a decomposition

$$\mathcal{B}_g(M) = \bigoplus_{i=1}^r K[a^i, b^i]$$

of $\mathcal{B}_g(M)$ as a direct sum of subquotients of R bounded in the interval $[0, g]$ and seen as K-vector spaces. Since $\mathcal{P}_g$ is a K-linear functor, by Proposition 2.4 we obtain the decomposition

$$M = \mathcal{P}_g(\mathcal{B}_g(M)) = \bigoplus_{i=1}^r \mathcal{P}_g(K[a^i, b^i]).$$

By Lemma 3.2, we obtain the desired decomposition [$\star$]. The statement about the Hilbert depth of $\mathfrak{D}(\mathfrak{P})$ follows straight from the definitions. This proves the statement (1).

(2) Let $T = K[Z](-a)$ be a Hilbert space. Then we have

$$\mathcal{B}_g(T) = \begin{cases} K[a, b(a)] & \text{if } a \preceq g; \\ 0 & \text{otherwise,} \end{cases}$$

where the components of $b(a) \in \mathbb{N}^n$ are defined as

$$b(a)_j = \begin{cases} a_j & \text{if } X_j \notin Z; \\ g_j & \text{otherwise.} \end{cases}$$

In particular $\rho(b(a)) \geq |Z|$. Therefore, if $\mathfrak{D} : M = \bigoplus_{i=1}^r K[Z_i](-a^i)$ is a Hilbert decomposition of M, then

$$\mathcal{B}_g(M) = \bigoplus_{i=1}^{r} \mathcal{B}_g(K[Z_i](-a^i)) = \bigoplus_{\substack{i \\ a^i \preceq g}} K[a^i, b(a)^i]$$

where $\rho(b(a)^i) \geq |Z_i|$ for all i such that $a^i \preceq g$. It follows that

$$\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i=1}^{r} Q[a^i, b(a)^i](X)$$

is a Hilbert partition of $H_M(X)_{\preceq g}$, and (1) implies that $\operatorname{Hdepth} \mathfrak{D}(\mathfrak{P}) \geq \operatorname{Hdepth} \mathfrak{D}$. $\square$

It is now an easy matter to check:

Corollary 3.4. *Let M a finitely generated multigraded R-module. Then*

$$\operatorname{Hdepth} M = \max\{\operatorname{Hdepth} \mathfrak{D}(\mathfrak{P}) : \mathfrak{P} \textit{ is a Hilbert partition of } H_M(X)_{\preceq g}\}.$$

In particular, there exists a Hilbert partition $\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i=1}^r Q[a^i, b^i](X)$ of $H_M(X)_{\preceq g}$ such that

$$\operatorname{Hdepth} M = \min\{\rho(b^i) : \ i = 1, \ldots, r\}.$$

We finish this section with two examples which show that Corollary 3.4 can be used in an effective way for computing $\operatorname{Hdepth} M$.

Example 3.5. Let $R = K[X_1, X_2]$ with $\deg(X_1) = (1, 0)$ and $\deg(X_2) = (0, 1)$. Let $M = R \oplus (X_1, X_2)R$. First of all, a minimal multigraded free resolution of M is obtained by adding a minimal multigraded free resolution of R

$$0 \longrightarrow R(-(0, 0)) \longrightarrow R(-(0, 0)) \longrightarrow 0$$

and a minimal multigraded free resolution of $(X_1, X_2)R$, namely

$$0 \longrightarrow R(-(1, 1)) \longrightarrow R(-(0, 1)) \oplus R(-(1, 0)) \longrightarrow M \longrightarrow 0.$$

This shows that we may choose $g = (1, 1)$. A simple inspection to the shape of M shows that

$$H_M(X_1, X_2)_{\preceq(1,1)} = 1 + 2X_1 + 2X_2 + 2X_1X_2.$$

It is easy to see that there are no Hilbert partitions containing only monomials of degree two as right ends of the intervals. The Hilbert partitions containing monomials of degree ≥ 1 as right ends of the intervals are

$$\begin{aligned}
\mathfrak{P}_1 :&(1 + X_1 + X_2 + X_1X_2) + (X_1 + X_1X_2) + X_2,\\
\mathfrak{P}_2 :&(1 + X_1 + X_2 + X_1X_2) + (X_2 + X_1X_2) + X_1,\\
\mathfrak{P}_3 :&(1 + X_1 + X_2 + X_1X_2) + X_1 + X_2 + X_1X_2,\\
\mathfrak{P}_4 :&(1 + X_1) + (X_1 + X_1X_2) + 2X_2 + X_1X_2,\\
\mathfrak{P}_5 :&(1 + X_1) + (X_1 + X_1X_2) + X_2 + (X_2 + X_1X_2),\\
\mathfrak{P}_6 :&(1 + X_1) + 2(X_2 + X_1X_2) + X_1,\\
\mathfrak{P}_7 :&(1 + X_1) + (X_2 + X_1X_2) + X_1 + X_2 + X_1X_2,\\
\mathfrak{P}_8 :&(1 + X_1) + X_1 + 2X_2 + 2X_1X_2,\\
\mathfrak{P}_9 :&(1 + X_2) + (X_2 + X_1X_2) + 2X_1 + X_1X_2,\\
\mathfrak{P}_{10} :&(1 + X_2) + 2(X_1 + X_1X_2) + X_2,\\
\mathfrak{P}_{11} :&(1 + X_2) + (X_1 + X_1X_2) + (X_2 + X_1X_2) + X_1,\\
\mathfrak{P}_{12} :&(1 + X_2) + (X_1 + X_1X_2) + X_1 + X_2 + X_1X_2,\\
\mathfrak{P}_{13} :&(1 + X_2) + 2X_1 + X_2 + 2X_1X_2.
\end{aligned}$$

We see also that $\operatorname{Hdepth}(M) = 1$. In the sequel we will focus on $\mathfrak{P}_1$ and $\mathfrak{P}_3$. They are represented in Figure 2 where the monomials are indicated by $\circ$, and the corresponding coefficients by numbers with an arrow pointing at circles.

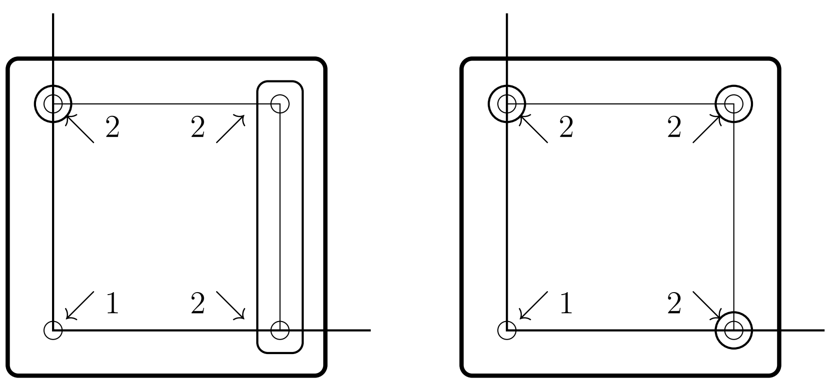

FIGURE 2. Hilbert partitions $\mathfrak{P}_1$ and $\mathfrak{P}_3$

Next we describe the induced Hilbert decompositions. For $\mathfrak{P}_1$ we have $r = 3$ with $[a^1, b^1] = [(0, 0), (1, 1)]$, $[a^2, b^2] = [(1, 0), (1, 1)]$ and $[a^3, b^3] = [(0, 1), (0, 1)]$, and so

$Z_{b^1} = Z_{b^2} = \{X_1, X_2\}$ and $Z_{b^3} = \{X_2\}$. Therefore

$$\begin{aligned}\mathcal{G}([(0,0),(1,1)]) =& \{(0,0)\},\\ \mathcal{G}([(1,0),(1,1)]) =& \{(1,0)\},\\ \mathcal{G}([(0,1),(0,1)]) =& \{(0,1)\}.\end{aligned}$$

The induced Hilbert decomposition of M is in this case

$$\mathfrak{D}(\mathfrak{P}_1) : M = K[X_1, X_2](-(0,0)) \oplus K[X_1, X_2](-(1,0)) \oplus K[X_2](-(0,1)).$$

Similarly one gets

$$\mathfrak{D}(\mathfrak{P}_3) : \quad M = \quad K[X_1, X_2](-(0,0)) \oplus K[X_1, X_2](-(1,1)) \oplus K[X_1](-(1,0)) \oplus K[X_2](-(0,1)).$$

Example 3.6. Let $R = K[X_1, X_2]$ and $M = K \oplus X_2K[X_2] \oplus X_2K[X_1, X_2] = R/(X_1, X_2) \oplus X_2R/(X_1) \oplus X_2R$. Similar arguments involving the graded free resolution of M show that one can choose $g = (1,1)$. Then

$$H_M(X_1, X_2)_{\preceq(1,1)} = 1 + 2X_2 + X_1X_2.$$

It is easily seen that there are no Hilbert partitions containing only monomials of degree two as right ends of the intervals and the Hilbert partitions containing monomials of degree ≥ 1 as right ends of the intervals are

$$\begin{aligned}\mathfrak{P}_1 :& (1 + X_2) + (X_2 + X_1X_2),\\ \mathfrak{P}_2 :& (1 + X_2) + X_2 + X_1X_2.\end{aligned}$$

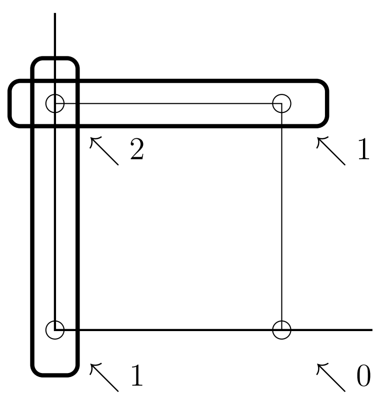

FIGURE 3. Name

They yield the induced Hilbert decompositions

$$\begin{aligned}\mathfrak{D}(\mathfrak{P}_1) :& M = K[X_2](-(0,0)) \oplus K[X_1, X_2](-(0,1)),\\ \mathfrak{D}(\mathfrak{P}_2) :& M = K[X_2](-(0,0)) \oplus K[X_2](-(0,1)) \oplus K[X_1, X_2](-(1,1)).\end{aligned}$$

Notice that Hdepth $\mathfrak{D}(\mathfrak{P}_1) =$ Hdepth $\mathfrak{D}(\mathfrak{P}_1) = 1$, and we have $\mathrm{Hdepth}(M) = 1$.

4. A method for computing the Stanley depth of a module

In this section we shall use Theorem 3.3 in order to compute the Stanley depth of a finitely generated multigraded R-module M. For simplicity we shall make the same assumptions as in Section 3. First of all we recall what Stanley depth is.

Definition 4.1. A *Stanley decomposition of* M is a finite family

$$\mathfrak{F} : (R_i, m_i)_{i\in I}$$

in which m_i are homogeneous elements of M and R_i are subalgebras generated by a subset of the indeterminates of R such that $R_i \bigcap \operatorname{Ann} m_i = 0$ for each $i \in I$, and

$$M \cong \bigoplus_{i\in I} m_i R_i$$

as a multigraded K-vector space. The multigraded K-subspace $m_i R_i \subset M$ is called a *Stanley space*.

Note that every Stanley decomposition induces a Hilbert decomposition and, in particular, it has a well defined depth. We shall use the following definition.

Definition 4.2. The *Stanley depth* of a module M is

$$\max\{\operatorname{depth} \mathfrak{F} \mid \mathfrak{F} \text{ is a Stanley decomposition of } M\}$$

and will be denoted by $\operatorname{Stdepth} M$.

Remark 4.3. Notice that Stanley depth equals Hilbert depth if $\dim_K M_a \le 1$ for all $a \in \mathbb{N}^n$ and $R_s M_t \neq 0$ whenever $R_s, M_t, M_{s+t} \neq 0$ (see [6, Proposition 2.8]). This is true for example if $M = I/J$ where $J \subset I$ are two monomial ideals. In this particular case, Theorem 2.1 in [11] provides a method to compute $\operatorname{Stdepth} M = \operatorname{Hdepth} M$. Theorem 3.3 may be seen as a natural extension of the method presented in [11] to the general situation of a finitely generated multigraded R-module M.

The fact that Theorem 2.1 in [11] does not extend naturally to a method for computing Stanley depth (but rather to a method for computing Hilbert depth) explains the well-known difficulty to compute Stanley depth of a finitely generated multigraded R-module M. Note that Theorem 3.3 can be always used directly for computing Stanley depth when $\operatorname{Stdepth} M = \operatorname{Hdepth} M$ (like the situation described above).

In the sequel we present a possible approach to the problem of computing Stanley depth of a finitely generated multigraded R-module M. The next proposition shows that one can decide in a *finite* number of steps if a Hilbert decomposition is inducing a Stanley decomposition or not (compare with [6, Proposition 2.9]).

Proposition 4.4. *Let* $\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{i=1}^{r} Q[a^i, b^i](X)$ *be a Hilbert partition of* $H_M(X)_{\preceq g}$*, and let*

$$\mathfrak{D}(\mathfrak{P}) : M = \bigoplus_{i=1}^{r} \Big(\bigoplus_{c\in\mathcal{G}[a^i,b^i]} K[Z_{b^i}](-c) \Big) = \bigoplus_{i\in I_{\mathfrak{P}}} R_i(-s_i)$$

be the induced Hilbert decomposition of M (note that $I_{\mathfrak{P}}$ is finite, since it depends on the Hilbert partition $\mathfrak{P}$, and $s_i \preceq g$). For all $i \in I_{\mathfrak{P}}$, choose $0 \neq m_i \in M_{s_i}$. The following statements are equivalent:

(1) *The decomposition*
$$M \cong \bigoplus_{i \in I_{\mathfrak{P}}} m_i R_i$$
is a Stanley decomposition of M.

(2) *For all $i \in I_{\mathfrak{P}}$ we have that $R_i \bigcap \operatorname{Ann} m_i = 0$, and if*
$$\sum_{i \in I_{\mathfrak{P}}} m_i \Big(\sum_{s_i + t_{i_j} \preceq g} \alpha_{i_j} X^{t_{i_j}} \Big) = 0$$
with $\alpha_{i_j} \in K$, $X^{t_{i_j}} \in R_i$, then $\alpha_{i_j} = 0$ for all i_j.

All the Stanley decompositions induced by suitable choices of elements m_i have the same Stdepth *equal to* $\operatorname{Hdepth} \mathfrak{D}(\mathfrak{P})$.

Proof. We only have to show that (2) implies (1). The condition $R_i \bigcap \operatorname{Ann} m_i = 0$ assures that $m_i R_i$ is a Stanley space. In order to prove that the sum in (1) is direct, it suffices to show that any two different Stanley spaces in (1) have no homogeneous element in common.

Let $m_s \in M_s$ be a homogeneous element and assume for simplicity that
$$m_s \in m_{s_1} K[Z_{b^1}] \cap m_{s_2} K[Z_{b^2}],$$
where $m_{s_1} \in M_{s_1}$, $m_{s_2} \in M_{s_2}$ and $s_1, s_2 \in I_{\mathfrak{P}}$. It is clear that $s_1 \preceq s$ and $s_2 \preceq s$, and therefore
$$m_{s_1} \alpha_1 X^{t_1} = m_{s_2} \alpha_2 X^{t_2},$$
where $s_1 + t_1 = s_2 + t_2 = s$, $\alpha_1, \alpha_2 \in K$, $X^{t_1} \in K[Z_{b^1}]$ and $X^{t_2} \in K[Z_{b^2}]$. If $s \preceq g$ then (2) implies directly $\alpha_1 = \alpha_2 = 0$.

Next let us suppose $s \not\preceq g$. We have $s_1 \preceq s$ and $s_1 \preceq g$, which implies $s_1 \preceq s \wedge g$. We claim $X^{s - s\wedge g} \in K[Z_{b^1}]$. If $b^1_l = g_l$ then $X_l \in Z_{b^1}$. If $b^1_l < g_l$ then $X_l \notin Z_{b^1}$ and hence $s_l = (s_1)_l \leq g_l$. It follows $(s \wedge g)_l = s_l$ and hence $(s - s \wedge g)_l = 0$.

Similarly we have $X^{s - s\wedge g} \in K[Z_{b^2}]$. Since M is positively g-determined, the multiplication map
$$\cdot X^{s - s\wedge g} : M_{s \wedge g} \longrightarrow M_s$$
is an isomorphism. Hence
$$m_{s_1} \alpha_1 X^{t_1 - s + s\wedge g} = m_{s_2} \alpha_2 X^{t_2 - s + s \wedge g}.$$
Now it is easily seen that
$$s_1 + t_1 - s + s \wedge g = s_2 + t_2 - s + s \wedge g = s \wedge g \preceq g$$
and (2) implies $\alpha_1 = \alpha_2 = 0$. $\square$

Remark that in general a Hilbert partition will not induce a Stanley decomposition, as the following example shows.

Example 4.5. Let us consider again the module

$$M = K \oplus X_2K[X_2] \oplus X_2K[X_1, X_2] = R/(X_1, X_2) \oplus X_2R/(X_1) \oplus X_2R$$

of Example 3.6. Then $K[X_2](-(0,0)) \oplus K[X_1, X_2](-(0,1))$ is a Hilbert decomposition of M which does not induce a Stanley decomposition $M = m_1K[X_2] \oplus m_2K[X_1, X_2]$. Since $M_{(0,0)} = K$ and every element in K is annihilated by the ideal (X_1, X_2), there is no possible choice for m_1. The same holds for the Hilbert decomposition $K[X_2](-(0,0)) \oplus K[X_2](-(0,1)) \oplus K[X_1, X_2](-(1,1))$. We conclude that $\operatorname{Stdepth} M = 0$.

Proposition 4.4 allows us to prove the main result of this section, which shows that the Stanley depth can be computed by looking at the Hilbert partitions.

Theorem 4.6. *Let $\mathfrak{F}$ be a Stanley decomposition of M. Then there exists a Hilbert partition $\mathfrak{P}$ of $H_M(X)_{\preceq g}$ inducing a Hilbert decomposition*

$$\mathfrak{D}(\mathfrak{P}) : M = \bigoplus_{i \in I_{\mathfrak{P}}} R_i(-s_i)$$

and $0 \neq m_i \in M_{s_i}$ for all $i \in I_{\mathfrak{P}}$, such that the Hilbert decomposition $\mathfrak{D}(\mathfrak{P})$ induces a Stanley decomposition

$$\overline{\mathfrak{D}(\mathfrak{P})} : M \cong \bigoplus_{i \in I_{\mathfrak{P}}} m_i R_i$$

with $\operatorname{Stdepth} \overline{\mathfrak{D}(\mathfrak{P})} \geq \operatorname{Stdepth} \mathfrak{F}$.

Proof. Let $mK[Z]$ be a Stanley space in $\mathfrak{F}$ such that $m \in M_a$. Then we have

$$\mathcal{B}_g(mK[Z]) = \begin{cases} K[a, b(a)] & \text{if } a \preceq g; \\ 0 & \text{otherwise,} \end{cases}$$

where

$$b(a)_l = \begin{cases} a_l & \text{if } X_l \notin Z; \\ g_l & \text{otherwise.} \end{cases}$$

Suppose $a \preceq g$. Then $X_l \in Z_{b(a)}$ only if $X_l \in Z$ or $a_l = g_l$. We have

$$\operatorname{Ann} m \cap K[Z] = 0$$

since $mK[Z]$ is a Stanley space. If $a_l = g_l$ then $\operatorname{Ann} m \cap K[Z \cup \{X_l\}] = 0$ since M is positively g-determined, so the multiplication map $\cdot X_l : M_a \longrightarrow M_{a+e_l}$ is injective. We may replace Z by $Z \cup \{X_l\}$ and after a finite number of steps we deduce that

$$\operatorname{Ann} m \cap K[Z_{b(a)}] = 0.$$

Remark the following fact:

$(*)$: If $X^t \in K[Z_{b(a)}]$ and $a + t \preceq g$, then $X^t \in K[Z]$.

Let $\mathfrak{F} : M \cong \bigoplus_{i=1}^{r} m_iK[Z_i]$ with $m_i \in M_{a^i}$ be the Stanley decomposition of M. Then

$$\mathfrak{P} : H_M(X)_{\preceq g} = \sum_{a^i \preceq g} Q[a^i, b(a^i)](X)$$

is a Hilbert partition of $H_M(X)_{\preceq g}$ and $\operatorname{Ann} m_i \cap K[Z_{b(a^i)}] = 0$. Moreover, if

$$\sum_{a^i \preceq g} m_i \Big(\sum_{a^i + t_{i_j} \preceq g} \alpha_{i_j} X^{t_{i_j}} \Big) = 0$$

with $\alpha_{i_j} \in K$, $X^{t_{i_j}} \in K[Z_{b(a^i)}]$, then the fact $(*)$ implies that $X^{t_{i_j}} \in K[Z_i]$. It follows that $\alpha_{i_j} = 0$ for all i_j since $\mathfrak{F}$ is a Stanley decomposition. By Proposition 4.4 it is easily seen that the induced decomposition

$$\overline{\mathfrak{D}(\mathfrak{P})} : \sum_{a^i \preceq g} m_i K[Z_{b(a^i)}]$$

is a Stanley decomposition. Finally, Theorem 3.3 yields the desired inequality $\operatorname{Stdepth} \overline{\mathfrak{D}(\mathfrak{P})} \geq \operatorname{Stdepth} \mathfrak{F}$.

□

A procedure for the computation of the Stanley depth can be now simply deduced.

Corollary 4.7. Stdepth M *may be computed by considering the partitions* $\mathfrak{D}(\mathfrak{P})$*, where* $\mathfrak{P}$ *runs over the finitely many Hilbert partitions of* $H_M(X)_{\preceq g}$*, and selecting those for which there exist* $m_i \in M$ *for all* $i \in I_{\mathfrak{P}}$ *such that the condition (2) in Proposition 4.4 is fulfilled.*

The following example shows that Stanley decomposition induced by Hilbert partition can effectively be computed.

Example 4.8. We return to the Example 3.5. Let $M = R \oplus (X_1, X_2)R$. We consider again the Hilbert partitions

$$\begin{aligned} \mathfrak{P}_1 :& (1 + X_1 + X_2 + X_1X_2) + (X_1 + X_1X_2) + X_2, \\ \mathfrak{P}_3 :& (1 + X_1 + X_2 + X_1X_2) + X_1 + X_2 + X_1X_2. \end{aligned}$$

It is easy to check that

$$\begin{aligned} \overline{\mathfrak{D}(\mathfrak{P}_1)} : M =& (1,0)K[X_1,X_2] \oplus (0,X_1)K[X_1,X_2] \oplus (0,X_2)K[X_2], \\ \overline{\mathfrak{D}(\mathfrak{P}_3)} : M =& (1,0)K[X_1,X_2] \oplus (0,X_1X_2)K[X_1,X_2] \oplus (0,X_1)K[X_1] \oplus (0,X_2)K[X_2] \end{aligned}$$

are induced Stanley decompositions.

5. Some applications

As shown in the previous sections, both the Hilbert depth and the Stanley depth of a finitely generated multigraded R-module M can be computed by considering Hilbert partitions of the polynomial $H_M(X)_{\preceq g}$. Remark that these invariants can not be easily computed in practice, since the number of possible partitions is huge (even in very simple cases, see e.g. Example 3.5). In this section we will show that the methods introduced so far allow us however to deduce some simple statements. For simplicity we shall make the same assumptions as in Section 3.

The following proposition was proved in [11, Lemma 3.6] for ideals. Now we can state and prove it for Stanley depth of modules.

Proposition 5.1. *Let M be a finitely generated multigraded R-module. Let $R' = R \otimes_K K[X_{n+1}, \dots, X_{n+m}]$ be the polynomial ring in $n+m$ variables and $M' = M \otimes_K K[X_{n+1}, \dots, X_{n+m}]$ the module obtained from M by scalar extension. Then*

(1) $\operatorname{depth}_{R'} M' = \operatorname{depth}_R M + m$;
(2) $\operatorname{Hdepth}_{R'} M' = \operatorname{Hdepth}_R M + m$;
(3) $\operatorname{Stdepth}_{R'} M' = \operatorname{Stdepth}_R M + m$.

Proof. The statement about depth is clear since $X_{n+1}, \dots, X_{n+m}$ is a regular sequence for M'. Assume that the module M is positively g-determined and set $g' = (g, 0, \dots, 0) \in \mathbb{Z}^{n+m}$. Since the multiplication map $\cdot X_i : M_a \longrightarrow M_{a+e_i}$ is an isomorphism whenever $i \geq n+1$, we deduce that M' is positively g'-determined. It follows

$$H_M(X)_{\preceq g} = H_{M'}(X)_{\preceq g'} =: P(X),$$

from which we deduce the statement about Hdepth using Theorem 3.3 and Corollary 3.4. By Corollary 4.7 and Proposition 4.4 it is clear that a Hilbert partition of $P(X)$ is inducing a Stanley decomposition for M if and only if it is inducing a Stanley decomposition for M', and the last statement follows. □

In the same fashion as above, the following proposition—shown in [8, Lemma 2.2] for the case of Stanley depth of ideals—can be extended to modules.

Proposition 5.2. *Let $R' = R \otimes_K K[X_{n+1}, \dots, X_{n+m}]$ be the polynomial ring in $n+m$ variables and M' a finitely generated multigraded R"-module. We consider $\phi : R' \to R$, $\phi(X_i) = X_i$ for $i \leq n$ and $\phi(X_i) = 1$ for $n < i$. Let M be the R-module obtained from M' by scalar restriction via ϕ. Then*

(1) $\operatorname{depth}_{R'} M' \leq \operatorname{depth}_R M + m$;
(2) $\operatorname{Hdepth}_{R'} M' \leq \operatorname{Hdepth}_R M + m$;
(3) $\operatorname{Stdepth}_{R'} M' \leq \operatorname{Stdepth}_R M + m$.

Proof. The statement about depth is easy since every resolution of M' over R' induces a free resolution of M over R via ϕ, and we can then use the Auslander–Buchsbaum formula.

Further, remark that

$$H_M(X_1, \dots, X_n) = H_{M'}(X_1, \dots, X_n, 1, \dots, 1).$$

Assume that the module M' is positively $g' = (g_1, \dots, g_n, g_{n+1}, \dots, g_{m+n})$-determined and set $g = (g_1, \dots, g_n) \in \mathbb{Z}^n$. We deduce that M is positively g-determined and

$$H_M(X_1, \dots, X_n)_{\preceq g} = H_{M'}(X_1, \dots, X_n, 1, \dots, 1)_{\preceq g'}.$$

Given $a' = (a_1, \dots, a_n, a_{n+1}, \dots, a_{m+n}), b' = (b_1, \dots, b_n, b_{n+1}, \dots, b_{m+n}) \in \mathbb{Z}^{n+m}$ such that $a' \preceq b'$, we consider the polynomial induced by the interval $[a', b']$, namely

$$Q[a', b'](X) = \sum_{a' \preceq c \preceq b'} X^c.$$

It is easy to check that

$$Q[a',b'](X_1,\dots,X_n,1,\dots,1)=\Big(\prod_{n+1\le i\le n+m}(b_i-a_i+1)\Big)Q[a,b](X_1,\dots,X_n)$$

for $a=(a_1,\dots,a_n)$ and $b=(b_1,\dots,b_n)$. We conclude that each Hilbert partition of $H_{M'}(X)_{\preceq g'}$ with polynomials induced by intervals of type $[a',b']$ determinates a Hilbert partition of $H_M(X)_{\preceq g}$ with polynomials induced by intervals of type $[a,b]$, and the statement about Hdepth is deduced by using Theorem 3.3.

By Proposition 4.4 it is clear that a Hilbert partition of $H_{M'}(X)_{\preceq g'}$ is inducing a Stanley decomposition for M' only if it is inducing a Stanley decomposition for M', since there are less linear dependencies to check. The last statement on Stdepth follows straight. □

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